4.3 Permutations When all Objects are Distinguishable

Learning Targets:

1. Permutation formula: **nPr**
2. Understanding when to use the nPr permutation strategy
3. Solving a permutation counting problem:
   - using nPr
   - using nPr in a problem involving "**cases**"
   - using nPr in a problem involving **conditions or restrictions**
4. Comparing permutations created with and without **repetition**

The Permutation Formula:

When all n of the objects in a problem are being used in the arrangements, then there will be n! different permutations.

When you are only arranging some of the objects rather than all of them, there is a different formula.

The Permutation Formula:

If you have n distinguishable objects and you are creating permutations by using r of the objects at a time, then the number of permutations will be

$$ {}_{n}P_{r} = \frac{n!}{(n-r)!} \qquad (r \le n) $$

Example:

Suppose 5 people are in a group.

How many different ways can these 5 people be lined up in a row?

$$5! = 120$$

How many different ways can you line up ***3 of them at a time***?

$$_5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5 \cdot 4 \cdot 3 \cdot \cancel{2 \cdot 1}}{\cancel{2 \cdot 1}}$$

$$= 60$$

Evaluating ${}_{n}P_{r} = \frac{n!}{(n-r)!} \quad (r \leq n)$

$${}_{7}P_{3} = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{\cancel{4!}}$$

$$= 7 \cdot 6 \cdot 5 = 210$$

Evaluating ${}_{n}P_{r} = \frac{n!}{(n-r)!} \quad (r \le n)$

$$ {}_{8}P_{5} = \frac{8!}{(8-5)!} = \frac{8!}{3!} = $$

$$ = 6720 $$

Evaluating ${}_{n}P_{r} = \frac{n!}{(n-r)!} \quad (r \le n)$

$${}_{5}P_{1} = \frac{5!}{(5-1)!} = \frac{5!}{4!} = \frac{5 \cdot \cancel{4!}}{\cancel{4!}} = 5$$

Rule : ${}_{n}P_{1} = n$

Evaluating ${}_{n}P_{r} = \frac{n!}{(n-r)!} \quad (r \leq n)$

$$ {}_{5}P_{1} = \frac{5!}{(5-1)!} = \frac{5!}{4!} = \frac{5(4)(3)(2)(1)}{4(3)(2)(1)} = 5 $$

Rule: ${}_{n}P_{1} = n$

Examples (typical ${}_nP_r$ problems):

How many different displays can be made out of 6 model cars if only 4 of the cars can be used at a time?

n

r

$${}_6P_4 = 360$$

Examples (typical ${}_nP_r$ problems):

How many different ways can a President, Vice President and Secretary be chosen from a group of 12 students?

1 2 3

$\therefore r = 3$

n

$${}_{12}P_3 = 1320$$

Compare this with FCP:

$$\underset{P}{\underline{12}} \times \underset{VP}{\underline{11}} \times \underset{S}{\underline{10}} = 1320$$

Examples (typical $_nP_r$ problems):

How many three-letter arrangements are there of the letters in the word TRIANGLE?

(hint: imagine the letters are on Scrabble tiles)

$n = 8$

$r = 3$

$$_8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{\boxed{8 \cdot 7 \cdot 6} \cdot \cancel{5!}}{\cancel{5!}}$$

$$= 336$$

Examples (typical ${}_nP_r$ problems):

Matt recently downloaded 40 new songs. How many 12-song playlists can he create if no song gets repeated within a playlist?

$${}_{40}P_{12} = 2.676111756 \times 10^{18}$$

$0!$ is the only factorial where the number is not a counting number.

You can't "expand" 0! like other n! expressions so it is difficult to understand how 0! can have a value at all.

The expression ${}_nP_n$ would mean that you have a group of n objects and you were arranging n of them at a time. This is the same thing as n!

What does the ${}_nP_n$ permutation formula look like in its expanded form? What does it simplify to?

$$_{n}P_{n} = \frac{n!}{(n-n)!} = \frac{n!}{0!}$$

Now remember, this $_{n}P_{n}$ permuation formula means the same as n! so it should simplify to n!

Therefore, 0! must have a value of 1

(check this on your calculator)

Rule: ${}_nP_n = n!$

Rule: ${}_nP_0 = 1$

$= \frac{n!}{(n-0)!} = \frac{n!}{n!}$

$= 1$

Permutations involving "cases"

Look for an "OR" situation.

Sometimes the phrases "at least" or "at most" will be used.

Example:

Alex needs to create a password for her email account. The password can be made of digits from 0 to 9 and/or any letter of the alphabet. The password is also case-sensitive. The password must be at least 6 characters long, but no more than 8 characters long.

How many different passwords are possible if repetition of characters is not allowed?

Example:

Alex needs to create a password for her email account. The password can be made of digits from 0 to 9 and/or any letter of the alphabet. The password is also case-sensitive. The password must be at least 6 characters long, but no more than 8 characters long.

How many different passwords are possible if repetition of characters is not allowed?

What are the "cases"?

Example:

Alex's password could be 6 characters long <u>OR</u> it could be 7 characters long, <u>OR</u> it could be 8 characters long.

Count each case separately, then add the cases together.

The length of the password is the "r" value or each case. What is the "n" value for each case? $\}$ 62

6 characters: 7 characters: 8 characters:

$${}_{62}P_6 + {}_{62}P_7 + {}_{62}P_8$$

TOTAL: $\approx 1.388 \times 10^{14}$ different passwords

Permutations involving "conditions" or "restrictions":

Look for "if ... must" or "if... can't" situations.

Often will involve a <u>double strategy</u>: FCP where permutations will count the number of choices for each decision.

Example:

Jane has 12 different pairs of shoes and 4 different pairs of boots. In how many ways can Jane line up her footwear if the shoes must all be grouped together on the left, and the boots together on the right?

FCP: $\underbrace{12!}_{\text{Arrange the shoes}} \times \underbrace{4!}_{\text{Arrange the boots}} = 11,496,038,400$

Example:

Jane has 12 different pairs of shoes and 4 different pairs of boots. In how many ways can Jane line up her footwear if the shoes must all be grouped together on the left, and the boots together on the right?

Jane has to arrange her shoes AND arrange her boots (2 tasks).

______________ x ______________ =

arrange the shoes   arrange the boots

Comparing "repetition" versus "no repetition":

If we are creating arrangements of r objects taken from a group of n objects where repetition is allowed, the number of permutations is n^r.

n

r

Ex: How many $\boxed{3}$-letter passwords can be created using the 26 letters of the alphabet, allowing repetition of letters?

FCP: $\underset{\text{letter \#1}}{\underline{26}} \times \underset{\#2}{\underline{26}} \times \underset{\#3}{\underline{26}} = 26^3$

Comparing "repetition" versus "no repetition":

If we are creating arrangements of r objects taken from a group of n objects where repetition <u>is not</u> allowed, the number of permutations is $_{n}P_{r}$.

Ex: How many 3-letter passwords can be created using the 26 letters of the alphabet, if repetition of letters is not allowed?

$_{26}P_{3} =$

FCP: $\underset{1^{st}}{\underline{26}} \times \underset{2^{nd}}{\underline{25}} \times \underline{24} =$

Check your understanding:

pg. 255 - 256:

#1, 2, 3, 5, 7, 10, 12, 13