Lesson #8: Implicit Differentiation

(Section 4.10)

Learning Targets:

i) Calculating derivatives using implicit differentiation.

ii) Applying implicit differentiation to problems involving tangent line equations.

When a function is written in the form $y = f(x)$, we say that y is defined explicitly in terms of x.

Some functions and/or relations (such as circles) have equations that are defined **implicitly**, i.e. y is not isolated and cannot easily be isolated.

Ex. A circle centred at the origin with a radius of 6 has a equation $\boldsymbol{x^2 + y^2 = 36}$

We can still find the slope of a tangent line to such a relation using a process known as **implicit differentiation**, which employs a variation of the **chain rule**.

Ex.1 If $x^2 + y^2 = 25$ then :

a) Find $\frac{dy}{dx}$

b) Find the equation of the tangent line to the circleat the point $(-4,3)$.

$$x^2 + y^2 = 25$$

$$\frac{dx^2}{dx} + \frac{dy^2}{dx} = \frac{d(25)}{dx}$$

Differentiate each term with respect to "x"

This term requires the Chain Rule

$$\frac{dx^2}{dx} + \frac{dy^2}{dx} = \frac{d(25)}{dx}$$

$$2x + \frac{dy^2}{dy} \cdot \frac{dy}{dx} = 0$$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

Now we isolate dy/dx

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Now we isolate dy/dx

$$2y \cdot \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

b) Equation of the tangent line at (-4, 3)

For the slope, we evaluate dy/dx at
x = -4 and y = 3

$$slope = -\frac{x}{y} = -\frac{-4}{3} = \frac{4}{3}$$

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{4}{3}(x + 4)$$

$$y - 3 = \frac{4}{3}x + \frac{16}{3}$$

$$y = \frac{4}{3}x + \frac{16}{3} + \frac{9}{3}$$

$$y = \frac{4}{3}x + \frac{25}{3}$$

Implicit Differentiation Process

1. Differentiate each term of the equation with respect to x.
2. Collect the terms with dy/dx on one side of the equation.
3. Factor out the dy/dx.
4. Solve for dy/dx.

Ex.2 Use implicit differentiation to find *dy/dx* for the following:

$$2x^5 + x^4 y + y^5 = 36$$

This term will require the product rule where one function is the power of x and the other function is "y"

$$2x^5 + x^4y + y^5 = 36$$

$$10x^4 + \left(x^4 \cdot \frac{dy}{dx} + y \cdot 4x^3\right) + 5y^4 \cdot \frac{dy}{dx} = 0$$

$$\underbrace{10x^4 + 4x^3y} + \underbrace{x^4 \cdot \frac{dy}{dx} + 5y^4 \cdot \frac{dy}{dx}} = 0$$

Move the terms without dy/dx to the right side

Leave the dy/dx terms on the left

$$2x^5 + x^4y + y^5 = 36$$

$$10x^4 + \left(x^4 \cdot \frac{dy}{dx} + y \cdot 4x^3\right) + 5y^4 \cdot \frac{dy}{dx} = 0$$

$$10x^4 + 4x^3y + x^4 \cdot \frac{dy}{dx} + 5y^4 \cdot \frac{dy}{dx} = 0$$

$$x^4 \cdot \frac{dy}{dx} + 5y^4 \cdot \frac{dy}{dx} = -10x^4 - 4x^3y$$

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Factor out the dy/dx as a common factor on the left

$$2x^5 + x^4y + y^5 = 36$$

$$10x^4 + \left(x^4 \cdot \frac{dy}{dx} + y \cdot 4x^3\right) + 5y^4 \cdot \frac{dy}{dx} = 0$$

$$10x^4 + 4x^3y + x^4 \cdot \frac{dy}{dx} + 5y^4 \cdot \frac{dy}{dx} = 0$$

$$x^4 \cdot \frac{dy}{dx} + 5y^4 \cdot \frac{dy}{dx} = -10x^4 - 4x^3y$$

$$\frac{dy}{dx}(x^4 + 5y^4) = -10x^4 - 4x^3y$$

$$\frac{dy}{dx}(x^4 + 5y^4) = -10x^4 - 4x^3y$$

$$\frac{\frac{dy}{dx}(x^4 + 5y^4)}{(x^4 + 5y^4)} = \frac{-10x^4 - 4x^3y}{x^4 + 5y^4}$$

Divide to isolate dy/dx

$$\frac{dy}{dx} = \frac{-10x^4 - 4x^3y}{x^4 + 5y^4}$$

You Try:

Find dy/dx for the following:

$$x^2 + x^3y + y^2 = 16$$

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Find dy/dx for the following:

$$x^2 + x^3y + y^2 = 16$$

$$2x + x^3 \cdot \frac{dy}{dx} + y \cdot 3x^2 + 2y \cdot \frac{dy}{dx} = 0$$

$$2x + 3x^2y + x^3 \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} = 0$$

$$(x^3 + 2y)\frac{dy}{dx} = -2x - 3x^2y$$

$$\frac{dy}{dx} = \frac{-2x - 3x^2y}{x^3 + 2y}$$

Example #3

Find the slope of the tangent line at (2, 1):

$$x^3 - 4x^2y^4 + 3y = -5$$

$$3x^2 - \left(4x^2 \cdot 4y^3 \cdot \frac{dy}{dx} + y^4 \cdot 8x\right) + 3\frac{dy}{dx} = 0$$

$$3x^2 - 16x^2y^3 \cdot \frac{dy}{dx} - 8xy^4 + 3\frac{dy}{dx} = 0$$

$$(3 - 16x^2y^3)\frac{dy}{dx} = -3x^2 + 8xy^4$$

$$\frac{dy}{dx} = \frac{-3x^2 + 8xy^4}{3 - 16x^2y^3}$$

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Slope at (2, 1) $= \frac{-3(2)^2 + 8(2)(1)^4}{3 - 16(2)^2(1)^3}$

$$= \frac{-12 + 16}{3 - 64}$$

$$= -\frac{4}{61}$$

Assignment

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Written Exercises:
#3, 5, 8, 9, 11, 14, 16, 17, 20, 22, 23, 27, 30, 32