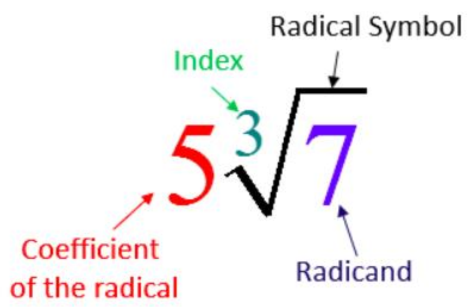
Pre-calculus 20
Chapter 5 Student Package

Parts of a Radical



If the index is not shown, it is implied to be 2.

Mixed radicals have a coefficient

Entire radicals have no coefficient

Prior Learning (*Math 10 Foundations and Pre-calculus*)

Integers that are perfect powers, have integer roots.

$\sqrt{16} = 4$ $\qquad$ $\sqrt[3]{27} = 3$ $\qquad$ $\sqrt[4]{625} = 5$

Some perfect powers and their roots are shown in the table:

Root	Perfect Powers			
x	x^2	x^3	x^4	x^5
1	1	1	1	1
2	4	8	16	32
3	9	27	81	243
4	16	64	256	
5	25	125	625	
6	36	216		
7	49	343		
8	64	512		
9	81	729		
10	100	1000		
11	121			
12	144			
13	169			
14	196			
15	225			
16	256			
17	289			
18	324			
19	361			
20	400			

Multiplication Property of Radicals: $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$

Examples: $\sqrt{24} = \sqrt{4} \cdot \sqrt{6}$

$\sqrt[3]{54} = \sqrt[3]{27} \cdot \sqrt[3]{2}$

This multiplication property allows us to simplify square roots, cube roots, fourth roots, etc. of numbers that aren't perfect powers, but do have ***factors*** that are perfect powers.

To write a radical of index ***n*** in simplest form, we write the radicand as a product of two factors, one of which is the **greatest *n*th power**. If the radicand does not contain any ***n***th power factors other than 1, then the radical is in simplest form.

Examples: $\sqrt{24} = \sqrt{4 \cdot 6} = \sqrt{4} \cdot \sqrt{6} = 2 \cdot \sqrt{6} = 2\sqrt{6}$ $\qquad$ * Note – we write our final answers

$\sqrt[3]{54} = \sqrt[3]{27 \cdot 2} = \sqrt[3]{27} \cdot \sqrt[3]{2} = 3 \cdot \sqrt[3]{2} = 3\sqrt[3]{2}$ $\qquad$ without the multiplication symbol

$\sqrt[4]{100}$ can't be simplified because none of the 4^{th} powers (except 1) are factors of 100

Example: **Many larger radicands could have more than one *n*th power factor**

$\sqrt{200} = \sqrt{\mathbf{4} \cdot 50}$ Which one of these three factorizations of 200

$\sqrt{200} = \sqrt{\mathbf{25} \cdot 8}$ contains the **"greatest *n*th power"** factor of 200?

$\sqrt{200} = \sqrt{\mathbf{100} \cdot 2}$ ← This one does (100 is the greatest perfect square factor of 200)

$\therefore \sqrt{200} = \sqrt{100 \cdot 2} = \sqrt{100} \cdot \sqrt{2} = 10\sqrt{2}$

Creating Entire Radicals:

To write a mixed radical as an entire radical, simply reverse the process we use to simplify a radical:

$4\sqrt{3} = \sqrt{16} \cdot \sqrt{3} = \sqrt{16 \cdot 3} = \sqrt{48}$ square root radical → write 4 as $\sqrt{16}$

$3\sqrt[3]{5} = \sqrt[3]{27} \cdot \sqrt[3]{5} = \sqrt[3]{27 \cdot 5} = \sqrt[3]{135}$ cube root radical → write 3 as $\sqrt[3]{27}$

$2\sqrt[4]{7} = \sqrt[4]{16} \cdot \sqrt[4]{7} = \sqrt[4]{16 \cdot 7} = \sqrt[4]{112}$ fourth root radical → write 2 as $\sqrt[4]{16}$

Section 5.1: Working with Radicals (2 days)

Key Ideas:

(1) What is meant by simplest form

(2) Simplifying mixed and entire radicals to simplest form

(3) Converting mixed radicals to entire radicals

(4) Radicands containing variable expressions

(5) Restrictions on variables

(6) Like radicals and unlike radicals

(7) Addition and subtraction of like radicals

Simplest form: a radical is in simplest form if the radicand does not contain a fraction or decimal, or any factor which may be removed, and the denominator does not contain a radical.

- "any factor which may be removed" refers to perfect power factors (*see chart on page 1*).
- radicands containing a fraction or decimal, and radicals in denominators will be dealt with in section 5.2

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Simplifying mixed and entire radicals – building on what we know:

How does the presence of a coefficient change the simplifying process?

Consider $\sqrt{75}$ and $4\sqrt{75}$. A coefficient of 4 means "multiply by 4" so $4\sqrt{75}$ simply means "4 times the square root of 75"

$\sqrt{75} = \sqrt{25 \cdot 3} = \sqrt{25} \cdot \sqrt{3} = 5\sqrt{3}$ this is what we already know

$4\sqrt{75} = 4\left(5\sqrt{3}\right) = \mathbf{20}\sqrt{3}$ the original coefficient multiplies onto the coefficient we get from simplifying the entire radical (it does not multiply the radicand)

Converting mixed radicals to entire radicals – building on what we know:

Anytime a radical has a coefficient of **either 1 or -1**, it is considered an entire radical. A coefficient of -1 simply shows up as a negative sign in front of the radical symbol.

When converting a mixed radical with a negative coefficient into an entire radical, we will **never put the negative inside the radical**. It will end up as a negative sign in front of the radical symbol of the entire radical:

$-3\sqrt{10} = -(3\sqrt{10}) = -1(\sqrt{9} \cdot \sqrt{10}) = -1\sqrt{9 \cdot 10} = -\sqrt{90}$

$-8\sqrt[3]{2} = -\sqrt[3]{512} \cdot \sqrt[3]{2} = -\sqrt[3]{512 \cdot 2} = -\sqrt[3]{1024}$

Radicals with variable expressions – building on what we know:

Perfect squares for variables will have exponents that are **multiples of 2**. When we take the square root of a perfect square variable, the exponent gets divided by 2.

$\sqrt{x^2} = x$ $\sqrt{a^6} = a^3$ $\sqrt{m^{16}} = m^8$ etc.

Perfect cubes for variables will have exponents that are **multiples of 3.** When we take the cube root of a perfect cube variable, the exponent gets divided by 3.

$\sqrt[3]{x^3} = x$ $\sqrt[3]{a^6} = a^2$ $\sqrt[3]{m^{18}} = m^6$ etc.

Perfect fourth powers for variables will have exponents that are **multiples of 4.** When we take the fourth root of a perfect fourth power variable, the exponent gets divided by 4.

$\sqrt[4]{x^4} = x$ $\sqrt[4]{a^{12}} = a^3$ $\sqrt[4]{m^{20}} = m^5$ etc.

Perfect fifth powers for variables will have exponents that are **multiples of 5.** When we take the fifth root of a perfect fifth power variable, the exponent gets divided by 5.

$\sqrt[5]{x^5} = x$ $\sqrt[5]{a^{15}} = a^3$ $\sqrt[5]{m^{20}} = m^4$ etc.

When a radical contains variables in its radicand, it is not in simplest form unless **every exponent is less than the index of the radical**. If the exponent is greater than or equal to the index, it can be simplified.

Simplifying radicals containing variable expressions will involve breaking down the variable expression as the product of two expressions, one of which is the expression with the highest exponent that is a perfect power for the index.

THIS PROCESS REQUIRES THE APPLICATION OF THE EXPONENT LAW FOR THE PRODUCT OF POWERS WITH THE SAME BASE: $\boldsymbol{a^m \cdot a^n = a^{m+n}}$

Examples: $\sqrt{x^3} = \sqrt{x^2 \cdot x} = \sqrt{x^2} \cdot \sqrt{x} = x\sqrt{x}$

$\sqrt[3]{a^{11}} = \sqrt[3]{a^9 \cdot a^2} = \sqrt[3]{a^9} \cdot \sqrt[3]{a^2} = a^3\sqrt[3]{a^2}$

The exponents on multiple variables get handled separately:

$\sqrt{\boldsymbol{m^5 n^4}}$ break down $\boldsymbol{m^5}$ as $\boldsymbol{m^4 \cdot m}$, $\boldsymbol{n^4}$ is a perfect square already so we leave it as is

$= \sqrt{\boldsymbol{m^4 \cdot m \cdot n^4}} = \sqrt{m^4} \cdot \sqrt{n^4} \cdot \sqrt{m} = m^2 \cdot n^2 \cdot \sqrt{m} = m^2n^2\sqrt{m}$

To create entire radicals from mixed radicals where the coefficient contains variables, the process is reversed:

$\mathbf{a^3}\sqrt{\mathrm{a}}$ to put the coefficient back in a square root radical, we must **DOUBLE** its exponent

$= \sqrt{\mathbf{a^6} \cdot \mathrm{a}}$ now add the exponents according to the exponent law

$= \sqrt{\mathrm{a}^7}$

$\mathbf{a}\sqrt[3]{\mathrm{a}^2}$ to put the coefficient back into a cube root radical, we must **TRIPLE** its exponent

$= \sqrt[3]{\mathbf{a^3} \cdot \mathrm{a}^2}$ now add the exponents according to the exponent law

$= \sqrt[3]{\mathrm{a}^5}$

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Restrictions on variables in radicands (restrictions occur for two reasons):

1) **To avoid having radicals that are undefined**

Undefined radicals occur when **even index** radicals have **negative radicand** values.

Remember: you can't find the square root of a negative number (or the fourth root, sixth root, etc)

Therefore, expressions such as $\sqrt{x}$ and $\sqrt[4]{x^3}$ are only valid if the value of x is not negative. In general, any even index radical that contains variables with odd exponents will have such a restriction.

Restriction: $\boldsymbol{x \geq 0}$ *(x must be greater than or equal to zero)*

Odd index radicals are allowed to have negative radicands. Therefore, there are **no such restrictions** on variables in the radicands of odd index radicals.

2) **To maintain the equality between a given expression and its reduced or simplified form**

Entire radicals with no negative sign in front must maintain a positive value after being simplified or reduced.

Thus $\sqrt{x^2} = x$ is really only true if x is not negative → Restriction: $\boldsymbol{x \geq 0}$

And $\sqrt{x^6} = x^3$ is really only true if x is not negative → Restriction: $\boldsymbol{x \geq 0}$

But $\sqrt{x^4} = x^2$ would be true for any value of x, so there would be no restriction

Summary for Restrictions:

a) Restrictions will only apply to even index radicals, not odd index radicals.
b) If an even index radical in simplest form has any variable with an odd exponent in the radicand, the variable has the restriction of "must be greater than or equal to zero".
c) If any variable has an even exponent as part of an unsimplified radicand, but an odd exponent when reduced or simplified, the variable must be greater than or equal to zero.
d) If any variable has an even exponent as part of an unsimplified radicand, and an even exponent when reduced or simplified, there will be no restriction on that variable.

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Like radicals and unlike radicals:

Two radicals are considered to be "like radicals" if they have the same index and the same radicand (after being reduced to simplest form). Radicals with different index numbers are always unlike radicals. Radicals with different radicands (after being reduced to simplest form) are considered unlike radicals.

Like radicals: $4\sqrt{2}$ and $-7\sqrt{2}$ $\quad$ $\sqrt[3]{15}$ and $6\sqrt[3]{15}$

Unlike radicals $\sqrt[3]{7}$ and $\sqrt{7}$ $\quad$ $\sqrt{6}$ and $\sqrt{10}$

Only "like radicals" can be combined using addition and subtraction.

Addition and subtraction of like radicals:

$$\boldsymbol{m\sqrt[r]{a} + n\sqrt[r]{a} = (m+n)\sqrt[r]{a}}$$
$$\boldsymbol{m\sqrt[r]{a} - n\sqrt[r]{a} = (m-n)\sqrt[r]{a}}$$

- combine their coefficients
- the radical remains unchanged

Example: $\sqrt[3]{4} + 5\sqrt[3]{4} = (1+5)\sqrt[3]{4} = 6\sqrt[3]{4}$

$4\sqrt{11} - 19\sqrt{11} = (4-19)\sqrt{11} = -15\sqrt{11}$