Lesson #5: Concavity and the Second Derivative Test (day 1)

(Section 5.4)

Learning Targets:

i) When is a function "**concave up**" or "**concave down**"?

ii) What are **inflection points**?

iii) What is the **Test for Concavity?**

Recall...

A function is said to be **increasing** if the graph is **"going uphill".**

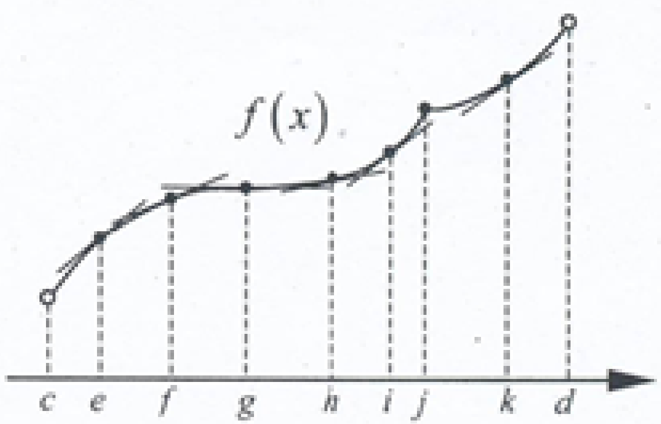


Figure 3

If $f'(x) > 0$ for all x in an open interval, then $f(x)$ is increasing on this open interval.

A function is said to be **decreasing** if the graph is **"going downhill".**

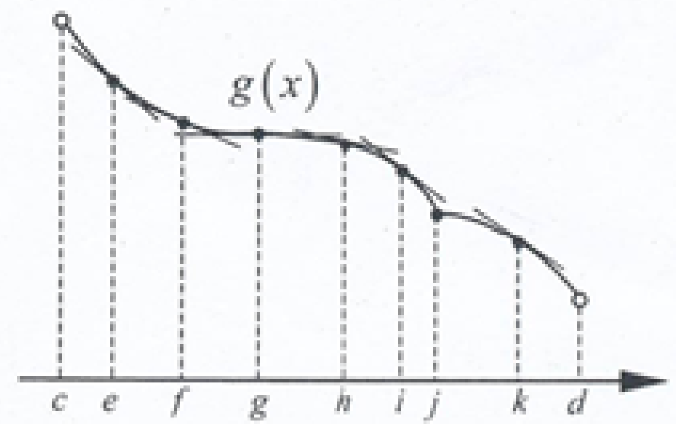


Figure 4

If $f'(x) < 0$ for all x in an open interval, then $f(x)$ is decreasing on this open interval.

Concave up – a function is said to be concave up if the graph is shaped like a "valley"

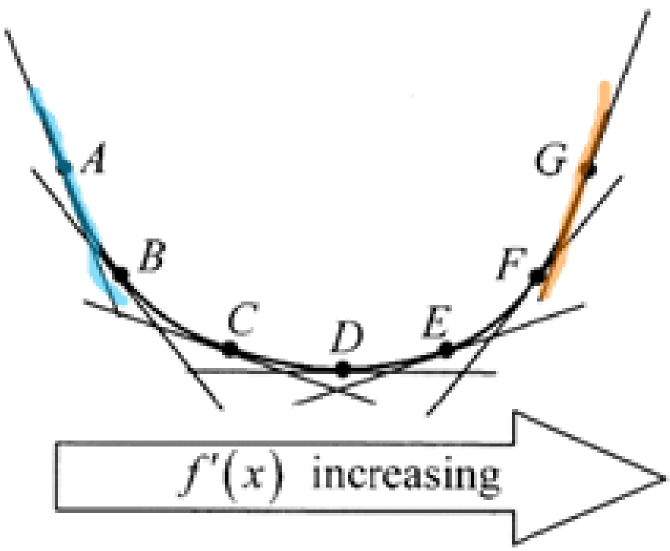


Figure 1
$f'(x)$ is increasing, thus $f''(x)$ is positive. The curve is concave up.

The slopes of the tangent lines **increase** from a large negative value at A (steeply slanted down) to a large positive value at G (steeply slanted up).

Between A and G the slopes of the tangent lines gradually become less steep negatives, then zero (at D), and then become more steep again as positives.

Concave up – a function is said to be concave up if the graph is shaped like a "valley"

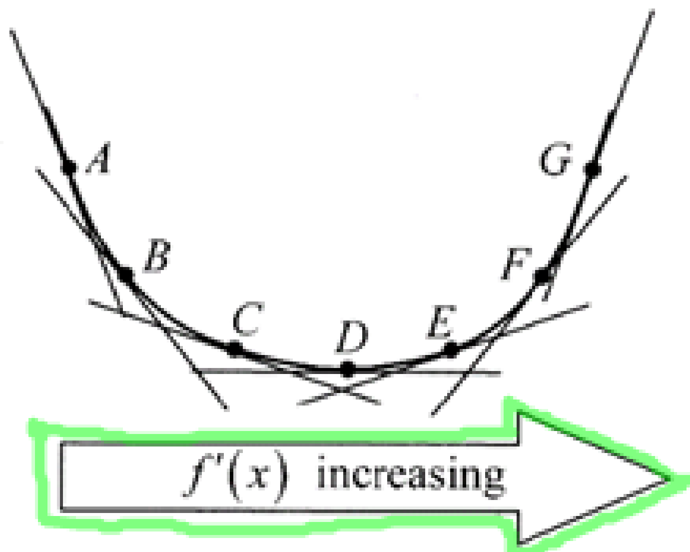


Figure 1

$f'(x)$ is increasing, thus $f''(x)$ is positive. The curve is concave up.

Just like f'(x) is positive when f(x) is increasing, f''(x) is positive when f'(x) is increasing.

Thus, f(x) is concave up on the intervals where f''(x) is positive.

Concave down – **a function is said to be concave down if the graph is shaped like a "hill"**

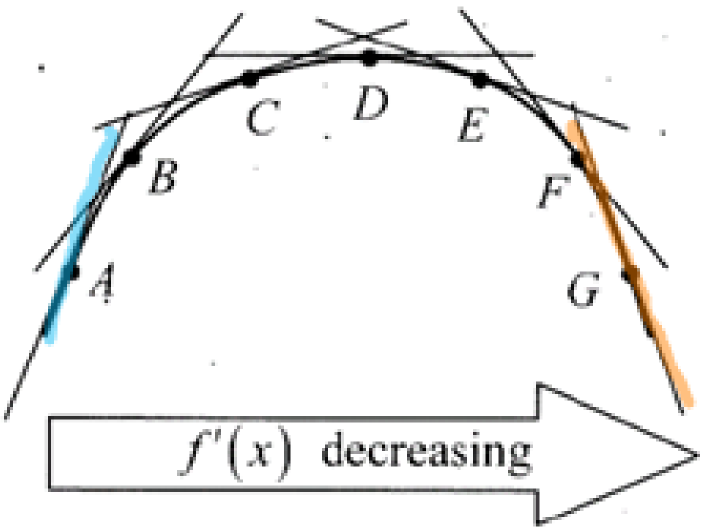


Figure 2

$f'(x)$ is decreasing, thus $f''(x)$ is negative. The curve is concave down.

The slopes of the tangent lines **decrease** from a large positive value at A (steeply slanted up) to a large negative value at G (steeply slanted down).

Between A and G the slopes of the tangent lines gradually become less steep positives, then zero (at D), and then become more steep again as negatives.

Concave down – a function is said to be concave down if the graph is shaped like a "hill"

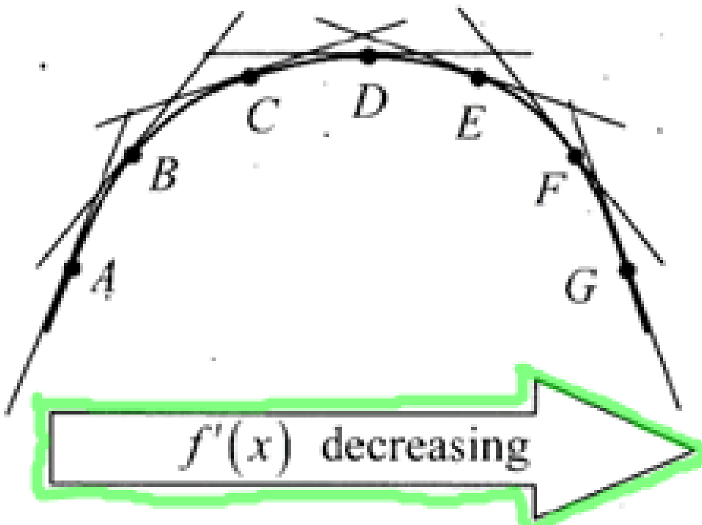


Figure 2

$f'(x)$ is decreasing, thus $f''(x)$ is negative. The curve is concave down.

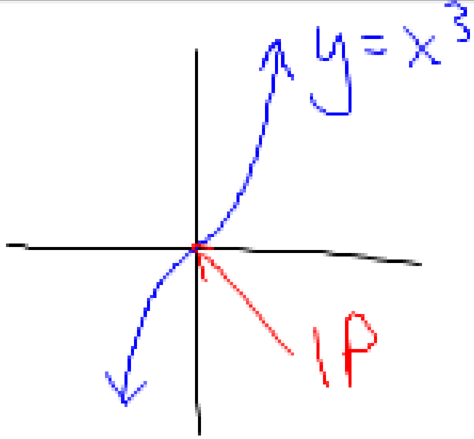
Just like f'(x) is negative when f(x) is decreasing, f''(x) is negative when f'(x) is decreasing.

Thus, f(x) is concave down on the intervals where f''(x) is negative.

Inflection Points

Inflection Points or IP's are points where the concavity of a function changes.

If inflection points exist, they will occur where $f''(x) = 0$ or where $f''(x)$ does not exist.



The Test for Concavity:

If $f''(x) > 0$ for all x in a particular interval, then $f(x)$ is concave up on that interval.

If $f''(x) < 0$ for all x in a particular interval, then $f(x)$ is concave down on that interval.

Therefore, we will perform a sign analysis on f"(x) to determine intervals of concavity.

Ex.1 Determine where the following functions are concave up, concave down and find all IP's

a) $y = x^3 - 3x^2 + 4x - 5$

① Find $f'(x)$
② Find $f''(x)$
③ $f''(x) = 0$
f'' is und.
④ Sign analysis on f''

$y' = 3x^2 - 6x + 4$

$y'' = 6x - 6$

$f'' = 0$

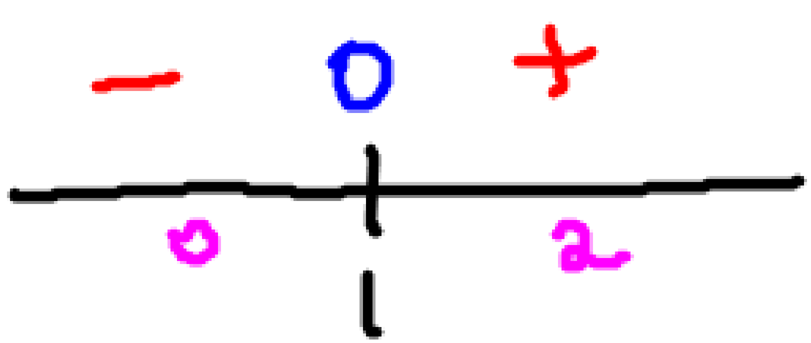
$6x - 6 = 0$

$6x = 6$

$x = 1$

f'' und

n/a



$f''(x) = 6x - 6$

f(x) is concave up on $(1, \infty)$

f(x) is concave down on $(-\infty, 1)$

IP will occur at x = 1

$$f(1) = 1^3 - 3(1)^2 + 4(1) - 5$$
$$f(1) = 1 - 3 + 4 - 5$$
$$f(1) = -3$$

$(1, -3)$

b) $y = \frac{x}{x^2+1}$

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2}$$

$$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2}$$

$$f'(x) = \frac{1-x^2}{(x^2+1)^2}$$

$$f'(x) = \frac{1-x^2}{(x^2+1)^2}$$

$$f''(x) = \frac{(x^2+1)^2(-2x) - (1-x^2)(2(x^2+1))(2x)}{(x^2+1)^4}$$

$$f''(x) = \frac{-2x(x^2+1)\left[x^2+1+(1-x^2)(2)\right]}{(x^2+1)^4}$$

$$f''(x) = \frac{-2x\cancel{(x^2+1)}(3-x^2)}{(x^2+1)^{\cancel{4}3}}$$

$$f''(x) = \frac{-2x(3-x^2)}{(x^2+1)^3}$$

$f''(x) = 0$	$f''(x)$ undefined
$-2x(3 - x^2) = 0$	n/a
	$(x^2 + 1 \neq 0)$

$-2x = 0$

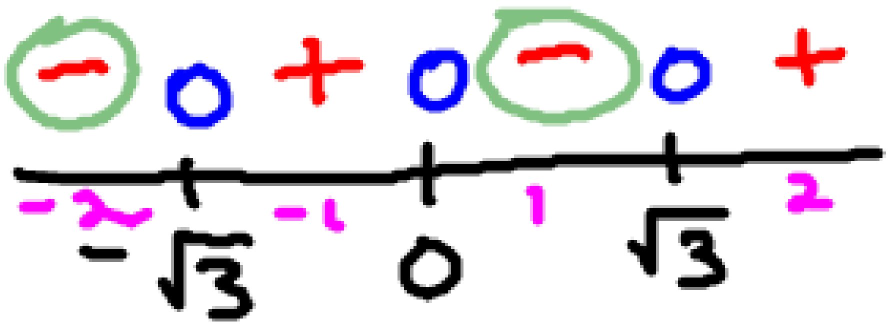
$x = 0$

$3 - x^2 = 0$

$3 = x^2$

$\pm\sqrt{3} = x$

Sign analysis on f''(x):



$$f''(x) = \frac{-2x(3-x^2)}{(x^2+1)^3}$$

always positive

Concave up $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

Concave down $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

Inflection points: $y = \dfrac{x}{x^2+1}$

$x = -\sqrt{3}$ $\quad y = \dfrac{-\sqrt{3}}{(-\sqrt{3})^2+1}$

$y = -\dfrac{\sqrt{3}}{4}$ $\quad \left(-\sqrt{3}, -\sqrt{3}/4\right)$

$x = 0$ $\quad y = \dfrac{0}{0^2+1}$

$y = 0$ $\quad (0, 0)$

$x = \sqrt{3}$ $\quad y = \dfrac{\sqrt{3}}{\sqrt{3}^2+1}$

$y = \sqrt{3}/4$ $\quad \left(\sqrt{3}, \sqrt{3}/4\right)$

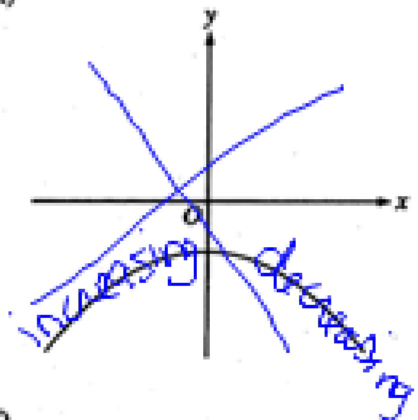
$f(x) < 0$
$\Rightarrow$ below x-axis

$f'(x) < 0$
$\Rightarrow f(x)$ is always decreasing

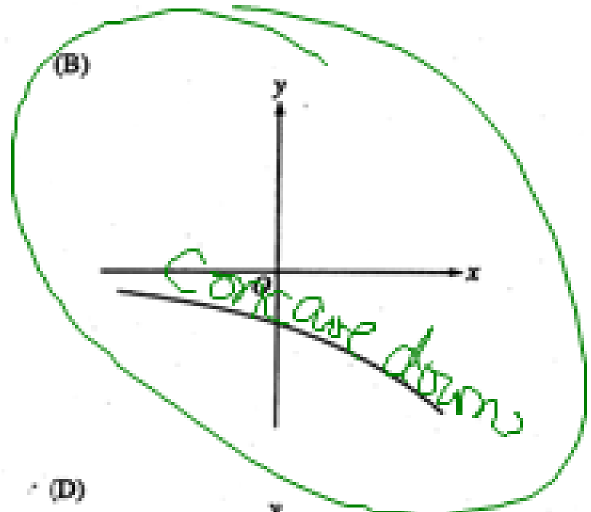
$f'' < 0$
$\Rightarrow f(x)$ is concave down

10. The function f has the property that $f(x)$, $f'(x)$, and $f''(x)$ are negative for all real values x. Which of the following could be the graph of f ?

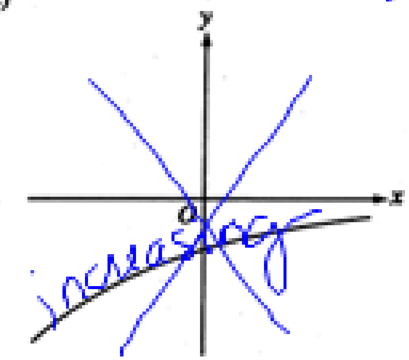
(A)



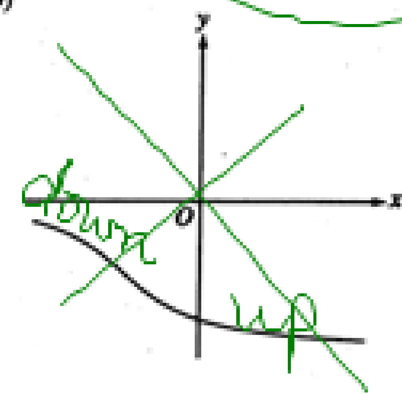
(B)



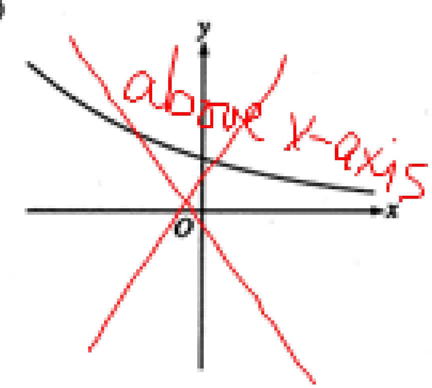
(C)



(D)



(E)



Assignment

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Written Exercises: #1(a)-(f), 4, 5, 6, 8, 11, 12, 14